Chapter 11

The Inclusion-Exclusion Principle

11.1 Statement and proof of the principle

We have seen the sum principle that states that for n pairwise disjoint sets $A_1, A_2, \ldots, A_n$,

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| \,.$$

What happens when the sets are not pairwise disjoint? We can still say something. Namely, the sum $\sum_{i=1}^{n} |A_i|$ counts every element of $\bigcup_{i=1}^{n} A_i$ at least once, and thus even with no information about the sets we can still assert that

$$\left| \bigcup_{i=1}^{n} A_i \right| \leq \sum_{i=1}^{n} |A_i| \,.$$

However, with more information we can do better. For a concrete example, consider a group of people, 10 of whom speak English, 8 speak French, and 6 speak both languages. How many people are in the group? We can sum the number of English- and French-speakers, getting $10 + 8 = 18$. Clearly, the bilinguals were counted twice, so we need to subtract their number, getting the final answer $18 - 6 = 12$. This argument can be carried out essentially verbatim in a completely general setting, yielding the following formula:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

What if there are three sets? Suppose in addition to the above English and French speakers, we have 14 German-language enthusiasts, among which 8 also speak English, 5 speak French, and 2 speak all three languages. How many people are there now? We can reason as follows: The sum $10 + 8 + 14 = 32$ counts the people speaking two languages twice, so we should subtract their number, getting $32 - 6 - 8 - 5 = 13$. But now the trilinguals have not been counted: They were counted three times in the first sum, and then subtracted three times as part of the bilinguals. So the final answer is obtained by adding their number: $13 + 2 = 15$. In general,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

In the case of arbitrarily many sets we obtain the inclusion-exclusion principle:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} \left|A_{i_1} \cap A_{i_2} \cap \ldots A_{i_k}\right|.$$

Proof. Each element in $\bigcup_{i=1}^{n} A_i$ is counted exactly once on the left side of the formula. Consider such an element a and let the number of sets A_i that contain a be j. Then a is counted

$$\binom{j}{1} - \binom{j}{2} + \ldots + (-1)^{j-1}\binom{j}{j}$$

times on the right side. But recall from our exploration of binomial coefficients that

$$\sum_{i=0}^{j}(-1)^i\binom{j}{i} = \sum_{i=0}^{j}(-1)^{i-1}\binom{j}{i} = -1 + \sum_{i=1}^{j}(-1)^{i-1}\binom{j}{i} = 0,$$

which implies

$$\binom{j}{1} - \binom{j}{2} + \ldots + (-1)^{j-1}\binom{j}{j} = 1,$$

meaning that a is counted exactly once on the right side as well. This establishes the inclusion-exclusion principle. □

11.2 Derangements

Given a set $A = \{a_1, a_2 \ldots, a_n\}$, we know that the number of bijections from A to itself is $n!$. How many such bijections are there that map no element $a \in A$ to itself? That is, how many bijections are there of the form $f : A \to A$, such that $f(a) \neq a$ for all $a \in A$. These are called *derangements*, or bijections with no *fixed points*.

We can reason as follows: Let S_i be the set of bijections that map the i-th element of A to itself. We are the looking for the quantity

$$n! - \left|\bigcup_{i=1}^{n} S_i\right|.$$

By the inclusion-exclusion principle, this is

$$n! - \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} \left|S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_k}\right|.$$

Consider an intersection $S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_k}$. Its elements are the permutations that map $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ to themselves. The number of such permutations is $(n-k)!$, hence $|S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_k}| = (n-k)!$. This allows expressing the number of derangements as

$$
\begin{aligned}
n! - \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (n-k)! &= n! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\end{aligned}
$$

Now, $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$ is the beginning of the Maclaurin series of e^{-1}. (No, you are not required to know this for the exam.) This means that as n gets larger, the number of derangements rapidly approaches $n!/e$. In particular, if we just pick a random permutation of a large set, the chance that it will have no fixed points is about $1/e$. Quite remarkable, isn't it!?